Matrix and Eigenvalue for EPPS 7370

Determinant and Inverse Matrix

When A is 2×2 matrix(square matrix)

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$a_{11}a_{22} - a_{12}a_{21}$ is called the determinant of matrix A. It is denoted by $|A|$.

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

For example,

$$A = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}, \qquad |A| = 4 \cdot 2 - 3 \cdot 1 = 5$$

Determinant is important to identify if a matrix has an inverse.
Number α has a unique number α^{-1} with the property that $\alpha\alpha^{-1} = 1$.
Matrix A can also have an inverse matrix A^{-1} such that AA^{-1} = I. However, every matrix does not have its inverse. There is a condition for having an inverse matrix.

$$A \textit{ has an inverse} \Longleftrightarrow |A| \neq 0$$

If $|A| = 0$, matrix A is said to be **singular**.

Hence, if

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad |A| = a_{11}a_{22} - a_{12}a_{21} \neq 0,$$

then

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Eigenvalue and Eigenvector

Suppose matrix A, vector x and scalar λ with the special property that

$$Ax = \lambda x.$$

This is very useful. We can easily deal with matrix A, but how about A^{100} ? It will be a big problem. However, $\quad Ax = \lambda x \quad \Longrightarrow \quad A^n x = \lambda^n x \quad$ can provide us with an easy way to deal with A^n. Here. x is called an eigenvector and the associated λ is called an eigenvalue. Now, how to find λ?

The formula to get λ is

$$|A - \lambda I| = 0$$

For example,

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$$

According to the formula,

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 3 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 6 = 0$$

We can get $\lambda_1 = -2$ and $\lambda_2 = 3$, which are the eigenvalues of A. Furthermore, we can get eigenvectors with these eigenvalues.

Time Series Application

Suppose an AR model,

$$\begin{pmatrix} 1 & -\phi_1 & -\phi_2 & \cdots \\ 0 & 1 & -\phi_1 & \cdots \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix} \begin{pmatrix} X_t \\ X_{t-1} \\ \vdots \end{pmatrix} = \begin{pmatrix} w_t \\ w_{t-1} \\ \vdots \end{pmatrix} \Longrightarrow \Phi X_t = w_t$$

In the matrix equation, we can rewrite Φ ;

$$\Phi = v\lambda v^{-1}$$

This is because

$$\Phi v = v\lambda \Longrightarrow \Phi = v\lambda v^{-1}$$

As explained above, we can obtain eigenvalues with the given formula.

Now, move the rewritten Φ to the RHS,

$$X_t = v^{-1}\lambda^{-1}v \ w_t$$

By definition of eigenvalue and eigenvector, stationarity of X_t depends on λ^{-1}. When the value is less than 1, X_t will be stationary. Therefore, we need to use the condition $|\frac{1}{\lambda_i}| < 1$ to check stationarity.

Reference :
Essential Mathematics for Economic Analysis
Further Mathematics for Economic Analysis